

All Multiparty Quantum States Can Be Made Monogamous

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We show that arbitrary multiparty quantum states can be made to satisfy monogamy by considering increasing functions of any bipartite quantum correlation that may itself lead to a non-monogamous feature. This is true for states of an arbitrary number of parties in arbitrary dimensions, and irrespective of whether the state is pure or mixed. The increasing function of the quantum correlation satisfies all the expected quantum correlation properties as the original one. We illustrate this by considering a thermodynamic quantum correlation measure, known as quantum work-deficit. We find that although quantum work-deficit is non-monogamous for certain three-qubit states, there exist polynomials of the measure that satisfy monogamy for those states.

I. INTRODUCTION

Sharing of quantum correlations among many parties is known to play an important role in quantum phenomena, ranging from quantum communication protocols [1–4] to cooperative events in quantum many-body systems [5, 6]. It is therefore important to conceptualize and quantify quantum correlations, for which investigations are usually pursued in two directions, viz. the entanglement-separability [7] and the information-theoretic [8] ones. Any such measure of quantum correlation is expected to satisfy a monotonicity (precisely, non-increasing) under an intuitively satisfactory set of local quantum operations.

In case the quantum state is shared between more than two parties, one also expects that all measures of quantum correlation would additionally follow a monogamy property [9–11], which restricts the sharability of quantum correlations among many parties. In the case of three parties, say, Alice, Bob and Charu, monogamy of a measure says that the sum of quantum correlations of the two-party local states between the Alice-Bob and the Alice-Charu pairs, should not exceed the quantum correlation of Alice with Bob and Charu taken together. Alice is therefore allotted a special status, and is called the “nodal observer”. The concept has also been carried over to more than two extra-nodal observers. Classical correlations certainly do not satisfy such a monogamy constraint. The monogamous nature of quantum correlations plays a key role in the security of quantum cryptography [12]. Surprisingly however, there are important and useful entanglement measures that do not satisfy monogamy for certain multiparty quantum states, an example being the entanglement of formation [13], which quantifies the amount of entanglement required for preparation of a given bipartite quantum state. Nevertheless, it was found that for three-qubit systems, the concurrence squared [14], a monotonically increasing function of the entanglement of formation is monogamous [9–11]. Recently, it was shown that the information-theoretic quantum correlation measure, quantum discord [15, 16], can violate monogamy [17–20] (cf. [21, 22]), and again a monotonically increasing function of the quantum

discord satisfies monogamy for three-qubit pure states [23].

In this paper, we show that if any bipartite quantum correlation measure, of an arbitrary number of parties in arbitrary dimensions, is non-increasing under loss of a part of a local subsystem, any multiparty quantum state is either already monogamous with respect to that measure or an increasing function of the bipartite measure can make it so. Note that the result holds for both pure and mixed states. It is interesting to note that the increasing function also satisfies all the properties for being a measure of quantum correlation, which include monotonicity under local operations and vanishing for “classically correlated” states (which is the set of separable states for measures of entanglement). To illustrate the result, we show that although the quantum work-deficit [24], an information-theoretic quantum correlation measure, violates monogamy even for three-qubit pure states, the states become monogamous when one considers integer powers of the measure. In stark contrast to what happens for concurrence and quantum discord, we show that for the three-qubit generalized W states [25, 26], the fourth power of quantum work-deficit is required to obtain monogamy for these states. In case of arbitrary three-qubit W-class states [25, 26] and the GHZ-class states [26, 27], to obtain monogamy of quantum work-deficit, one requires higher polynomials. We also find that three-qubit pure states that are monogamous with respect to quantum discord are also so with respect to quantum work-deficit.

In Sect. II, we prove the result about the transformability of all non-monogamous multiparty states into monogamous ones. We illustrate this result in the next section (Sect. III) by using the concept of quantum work-deficit, where we prove certain general results about monogamy of quantum work-deficit for arbitrary three-party quantum states. We present a conclusion in Sect. IV. A brief introduction to quantum work-deficit is given in Appendix A.

II. TURNING NON-MONOGAMOUS MULTISITE QUANTUM STATES INTO MONOGAMOUS ONES

In proving the results, we will work with three-party quantum states. However, they can easily be generalized to an arbitrary number of parties. Let \mathcal{Q} be a quantum correlation measure that is defined for arbitrary bipartite states (pure or mixed) in arbitrary dimensions. Consider a three-party quantum state (pure or mixed), ϱ_{ABC} , in arbitrary dimensions, shared between three observers, Alice (A), Bob (B), and Charu (C). Let \mathcal{Q}_{AB} denote the quantum correlation \mathcal{Q} for the two-party reduced state $\varrho_{AB} = \text{tr}_C \varrho_{ABC}$. \mathcal{Q}_{AC} is similarly defined. Let $\mathcal{Q}_{A:BC}$ denote the quantum correlation for the state ϱ_{ABC} in the $A : BC$ partition.

The measure \mathcal{Q} is said to satisfy monogamy for the state ϱ_{ABC} if $\mathcal{Q}_{A:BC} \geq \mathcal{Q}_{AB} + \mathcal{Q}_{AC}$. The idea is that a measure will be called monogamous for a certain shared quantum state if the amount of quantum correlations that Alice has with Bob and Charu separately would be smaller than what she has with her partners taken together. The measure will be called strictly monogamous for ϱ_{ABC} if $\mathcal{Q}_{A:BC} > \mathcal{Q}_{AB} + \mathcal{Q}_{AC}$. On the other hand, $\mathcal{Q}_{A:BC} < \mathcal{Q}_{AB} + \mathcal{Q}_{AC}$, will imply that the measure is non-monogamous for the corresponding state.

The following theorem demonstrates that the non-monogamous nature of any measure for any state can be transformed to a monogamous one (in fact, strictly so), by considering an increasing function of the measure. Let \mathcal{R} be the set of all real numbers.

Theorem 1: If \mathcal{Q} violates monogamy for an arbitrary three-party quantum state ϱ_{ABC} in arbitrary dimensions, there always exists an increasing function $f : \mathcal{R} \rightarrow \mathcal{R}$ such that

$$f(\mathcal{Q}_{A:BC}) > f(\mathcal{Q}_{AB}) + f(\mathcal{Q}_{AC}), \quad (1)$$

provided that \mathcal{Q} is monotonically decreasing under discarding systems and invariance under discarding systems occurs only for monogamy-satisfying states.

Proof: Let us first rename

$$\mathcal{Q}_{A:BC} = x, \quad \mathcal{Q}_{AB} = y, \quad \mathcal{Q}_{AC} = z,$$

for notational simplicity. Then the constraints in the premise of the theorem (non-monogamy and monotonicity of \mathcal{Q}) can be rewritten as

$$x < y + z, \quad x > y > 0, \quad x > z > 0.$$

Hence it follows that $0 < \frac{y}{x} < 1$ and $0 < \frac{z}{x} < 1$. This implies that

$$\lim_{n \rightarrow \infty} \left(\frac{y}{x}\right)^n = 0, \quad \lim_{n \rightarrow \infty} \left(\frac{z}{x}\right)^n = 0 \quad (2)$$

Hence $\forall \epsilon > 0$, there exists positive integers $n_1(\epsilon), n_2(\epsilon)$

such that

$$\begin{aligned} \left(\frac{y}{x}\right)^m &< \epsilon \quad \forall \text{ positive integers } m \geq n_1(\epsilon), \\ \left(\frac{z}{x}\right)^m &< \epsilon \quad \forall \text{ positive integers } m \geq n_2(\epsilon). \end{aligned} \quad (3)$$

Let us now choose $\epsilon = \epsilon_1 < \frac{1}{2}$. Therefore, $\left(\frac{y}{x}\right)^m < \epsilon_1$ and $\left(\frac{z}{x}\right)^m < \epsilon_1$, \forall positive integers $m \geq n(\epsilon_1)$, where $n(\epsilon_1) = \max\{n_1(\epsilon_1), n_2(\epsilon_1)\}$. Adding the inequalities, we have $\left(\frac{y}{x}\right)^m + \left(\frac{z}{x}\right)^m < 2\epsilon_1 < 1$, \forall positive integers $m \geq n(\epsilon_1)$. Hence the proof. ■

Note here that invariance under discarding part of a subsystem implying monogamy, holds for many quantum correlation measures, including entanglement of formation and concurrence for three-qubit systems and quantum discord in arbitrary-dimensional three-party states. Note also that the power of a measure vanishes for the same class of states for which the original measure vanishes, so that the set of states that is indicated to be “classical” by the original measure, is invariant after the transformation of the original measure into the new one. Let us also mention here that if a measure is monotonically non-increasing for a certain class of local operations (possibly assisted by classical communication between the parties), a positive integer power of the measure also has the same property. Note that while the cases of vanishing x, y, z have been ignored in the proof, they can be handled easily.

We now show that the class of monogamous states is closed under the operation of taking positive integral powers of the corresponding measure.

Theorem 2: If a quantum correlation measure is monogamous for a three-party quantum state, any positive integer power of the measure is also monogamous for the same state.

Proof: The premise implies that $x \geq y + z$. Then for any positive integer m , we have

$$x^m \geq (y + z)^m = \sum_{k=0}^m \binom{m}{k} y^k z^{m-k}, \quad (4)$$

which in turn is $\geq y^m + z^m$, as y, z are non-negative. Hence the proof. ■

III. ON MONOGAMY OF QUANTUM WORK-DEFICIT

We will now consider the monogamy properties of the information-theoretic quantum correlation measure, called quantum work-deficit (WD) [24], for arbitrary three-qubit pure states. In particular, this will help to illustrate that powers of a measure can lead to monogamous nature for a state, when the measure itself is not so.

We begin by relating the monogamy properties of quantum discord, quantum work deficit, and entanglement of formation. Consider an arbitrary pure

three-party state $|\psi\rangle_{ABC}$. Let us denote the quantum discord for the state $\sigma_{AB} = \text{tr}_C |\psi\rangle\langle\psi|$ by D_{AB} , where the measurement is performed by the observer B . D_{AC} is similarly defined, with the measurement being performed by the observer C . The entanglements of formation of σ_{AB} and σ_{AC} are denoted by E_{AB}^f and E_{AC}^f respectively. Similar notations are used for the different varieties of the quantum work-deficits, Δ , Δ^\leftarrow , and Δ^\rightarrow . See Appendix A for the definition of WD.

Proposition 1: For an arbitrary three-party pure state, $D_{AB} + D_{AC} + H(\{p_i^B\}) + H(\{p_j^C\}) = E_{AB}^f + E_{AC}^f + H(\{p_i^B\}) + H(\{p_j^C\}) \geq \Delta_{AB}^\leftarrow + \Delta_{AC}^\leftarrow \geq \Delta_{AB} + \Delta_{AC}$, where $H(\{p_i^B\})$ is the entropy produced by the measurement in B , and similarly for $H(\{p_j^C\})$.

Proof: It can be obtained from Ref. [10] that for an arbitrary pure state $|\psi\rangle_{ABC}$,

$$E_{AB}^f - \sum_i p_i^C S(I \otimes M_i \varrho_{AC} I \otimes M_i^\dagger / p_i^C) = 0, \quad (5)$$

where $\{M_i\}$ forms the optimal measurement by the observer C and p_i^C are the corresponding probabilities. Here $S(\cdot)$ denotes the von Neumann entropy of its argument. Therefore, $E_{AB}^f + H(\{p_i^C\}) - S(\sum_i I \otimes M_i \varrho_{AC} I \otimes M_i^\dagger) = 0$, where $H(\cdot)$ denotes the Shannon entropy of the probability distribution in its argument. Here we assume that projective measurements attain optimality, which is indeed the case for rank-2 states [28]. Consequently, $E_{AB}^f + H(\{p_i^C\}) \geq \Delta_{AB}^\leftarrow + S(\varrho_{AB}) \geq \Delta_{AB}^\leftarrow \geq \Delta_{AB}$. Hence the result. ■

Performing measurements on the first parties will lead to $2E_{BC}^f + H(\{p_i^A\}) + H(\{q_j^A\}) \geq \Delta_{AB}^\rightarrow + \Delta_{AC}^\rightarrow \geq \Delta_{AB} + \Delta_{AC}$, where $H(\{p_i^A\})$ ($H(\{q_j^A\})$) is the entropy produced in the measurement at A on σ_{AB} (σ_{AC}).

Theorem 3: For an arbitrary pure three-party quantum state $|\psi\rangle_{ABC}$, quantum discord is monogamous whenever the quantum work-deficit, Δ^\leftarrow , is so.

Proof: From the definitions of quantum discord and WD, we obtain

$$D_{AB} = S_B + \Delta_{AB} - H(\{p_i^B\}), \quad (6)$$

where S_B is the von Neumann entropy of $\sigma_B = \text{tr}_{AC} |\psi\rangle\langle\psi|$. Since $S_B - H(\{p_i^B\}) \leq 0$, $D_{AB} \leq \Delta_{AB}^\leftarrow$. For states for which WD is monogamous, we have

$$D_{AB} + D_{AC} \leq \Delta_{AB}^\leftarrow + \Delta_{AC}^\leftarrow \leq \Delta_{A:BC}^\leftarrow = S_A = D_{A:BC}. \quad (7)$$

Here we assume that the minimum of work-deficit and quantum discord are attained by the same measurement. It is easy to see that the theorem holds even if the first parties perform the measurements. ■

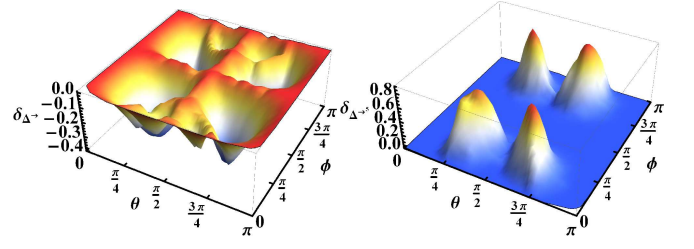


FIG. 1. Monogamy of quantum work-deficit. Left: The “monogamy score”, $\delta_{\Delta^\rightarrow}$, for quantum work-deficit is plotted, on the vertical axis, for the generalized W states, against the state parameters θ and ϕ on the base. Clearly, almost all generalized W states are non-monogamous with respect to quantum work-deficit as the quantum correlation measure. Right: All considerations are the same as in the left figure, except that the vertical axis represents the monogamy score $\delta_{(\Delta^\rightarrow)^5}$ corresponding to the fifth power of WD. As seen from the figure, almost all generalized W states are monogamous with respect to $(\Delta^\rightarrow)^5$. (The vertical axes in both the figures are in qubits, while the base axes are dimensionless for both.)

A. Monogamy of Work-deficit for W-class

We now consider the monogamy properties of quantum work-deficit for an important class of three-qubit pure states, viz. the generalized W states [25, 26], given by

$$|\phi_{GW}\rangle = \sin \theta \cos \phi |011\rangle + \sin \theta \sin \phi |101\rangle + \cos \theta |110\rangle, \quad (8)$$

where $\theta \in (0, \frac{\pi}{4}]$ and $\phi \in (0, 2\pi]$. We find that quantum work-deficit is non-monogamous for almost all members of this class (see Fig. 1 (left)). In other words, setting

$$\delta_Q(\varrho_{ABC}) \equiv \mathcal{Q}_{A:BC} - \mathcal{Q}_{AB} - \mathcal{Q}_{AC} \quad (9)$$

for an arbitrary bipartite quantum correlation measure \mathcal{Q} and an arbitrary three-party state ϱ_{ABC} , we find that

$$\delta_{\Delta^\rightarrow} < 0 \quad (10)$$

for about 98.97% of randomly chosen generalized W states. Note here that another information-theoretic quantum correlation measure, the quantum discord, can also be non-monogamous for these states [17–20]. However, recently it has been shown that the square of (one variety of) quantum discord is a monogamous quantity for all three-qubit pure states [23]. This however is no longer valid for WD. As stated in Theorem 1, suitably chosen integral powers of WD will be monogamous for any given state. And we find that for WD, monogamy for almost all generalized W states is obtained for the fifth power (see Fig. 1 (right)), i.e.

$$\delta_{(\Delta^\rightarrow)^5} > 0 \quad (11)$$

for about 99.72% of the generalized W states.

We have also considered the monogamy properties of general three-qubit pure states with respect to quantum work-deficit, Δ^\rightarrow . A histogram showing the relative

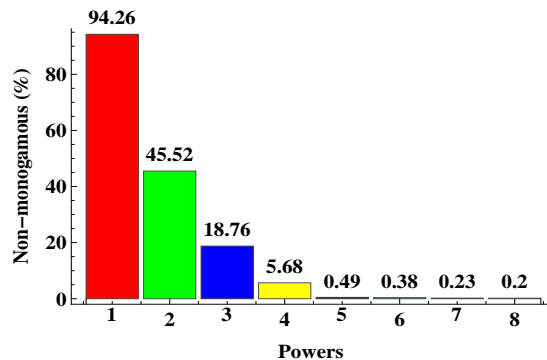


FIG. 2. Relative frequencies of non-monogamous three-qubit pure states. We provide estimates of the percentages of the complete space of three-qubit pure states with violates monogamy with respect to quantum work-deficit and its integral powers. The histogram in the figure shows the percentages on the vertical axis, while the different integral powers are on the horizontal axis. So, for example, the left-most (red) column indicates the estimated relative frequency of non-monogamous states with respect to the first power of WD. Both axes represent dimensionless parameters.

frequencies of non-monogamous states among randomly chosen pure three-qubit states, for different powers of quantum work-deficit, is given in Fig. 2.

IV. CONCLUSION

Quantum correlation measures can be monogamous or non-monogamous for multisite quantum states. This can happen for measures within the entanglement-separability paradigm, as well as those in the information-theoretic one. We demonstrated that any quantum correlation measure that is non-monogamous for a multipart quantum state can be made monogamous for the same by considering an increasing function of the measure. The transformed measure retains the important properties, like monotonicity under local operations and vanishing for “classical” states, of the original measure. We illustrate the results by using the concept of quantum work-deficit, an information-theoretic quantum correlation measure. We show that while the generalized W states are non-monogamous with respect to quantum work-deficit, the fourth power of the measure makes the states monogamous. We also discuss the monogamy properties of quantum work-deficit, and its powers, for arbitrary three-qubit pure states.

Let us mention here that in the literature, monotonically increasing functions of a quantum correlation measure is regarded with the same level of importance as the original measure. So, for example, the nearest-neighbor entanglement of spin-1/2 systems [5, 6] is usually investigated by employing the measure, concurrence, although

a more physically meaningful measure is the entanglement of formation, with concurrence being an increasing function of the latter.

ACKNOWLEDGMENTS

RP acknowledges an INSPIRE-faculty position at the Harish-Chandra Research Institute (HRI) from the Department of Science and Technology, Government of India, and SK thanks HRI for hospitality and support.

APPENDIX A: QUANTUM WORK-DEFICIT

In this appendix, we briefly introduce the information-theoretic measure of quantum correlation, known as quantum work-deficit [24] for an arbitrary bipartite quantum state ρ_{AB} . Let us begin by considering the number, I_G , of pure qubits that can be extracted from ρ_{AB} by “closed global operations”, with the latter consisting of any sequence of unitary operations and dephasing. It can be shown that

$$I_G(\rho_{AB}) = N - S(\rho_{AB}), \quad (12)$$

where N is the log of the dimension of the Hilbert space \mathcal{H} on which ρ_{AB} is defined. This thermodynamic “work” that can be extracted from the quantum state ρ_{AB} may require to employ global operations, which are not accessible to observers who are situated in separated laboratories. To obtain a quantification of the amount of work that can be extracted from ρ_{AB} by local actions, we restrict to “closed local quantum operations and classical communication (CLOCC)”, which consists of local unitaries, local dephasings, and sending dephased states from one party to another. Under these local actions, the number of pure qubits that can be extracted is given by

$$I_L(\rho_{AB}) = N - \inf_{\Lambda \in \text{CLOCC}} [S(\rho'_A) + S(\rho'_B)], \quad (13)$$

where $S(\rho'_A) = -\text{tr}_B(\Lambda(\rho_{AB})) \log \text{tr}_B(\Lambda(\rho_{AB}))$ and $S(\rho'_B) = -\text{tr}_A(\Lambda(\rho_{AB})) \log \text{tr}_A(\Lambda(\rho_{AB}))$. For an arbitrary bipartite state ρ_{AB} , the quantum work-deficit is then defined as

$$\Delta(\rho_{AB}) = I_G(\rho_{AB}) - I_L(\rho_{AB}), \quad (14)$$

and is interpreted as an information-theoretic quantum correlation measure of ρ_{AB} . The quantity is not efficiently computable for arbitrary bipartite states. General CLOCC actions are also difficult to implement in an experiment. Therefore we will also consider the quantity $\Delta_{AB}^{\rightarrow}$, in which we restrict our attention to CLOCC consisting of projection measurements at the single party (A) only for extracting work with local actions. If the measurement is performed by B , we denote it as Δ_{AB}^{\leftarrow} .

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